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Mixed convection in the stagnation point flow adjacent to a vertical surface in a viscoelastic fluid

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Abstract

An analytic technique, namely the homotopy analysis method (HAM), is applied to study the steady mixed convection in two-dimensional stagnation flows of a viscoelastic fluid around heated surfaces for the case when the temperature of the wall varies linearly with the distance from the stagnation point. The two-dimensional boundary layer equations governing the flow and thermal fields are reduced by appropriate transformations to a set of two ordinary differential equations. These equations are solved analytically using the HAM in the buoyancy assisting and opposing regions. It is found that, as for the Newtonian flow case, a reverse flow region develops in the buoyancy opposing flow region, and dual solutions are found to exist in that flow regime for a certain ranges of the buoyancy and viscoelastic parameters.
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Keywords: Stagnation point flow; Viscoelastic fluid; Heat transfer; Series solution; HAM

1. Introduction

The theory of non-Newtonian fluids has become a field of very active research for the last few decades as this class of fluids represents, mathematically, many industrially important fluids such as plastic films and artificial fibers in industry. Several authors have considered the viscoelastic fluids whose constitutive equations are based on the assumption of gradually fading memory (i.e. short relaxation times), see [1]. A good list of references on the published papers for these fluids can be found in [2–4]. The steady incompressible flow of a viscoelastic fluid in the region of a two-dimensional stagnation point flow has been studied by Beard and Walters [5] and Garg and Rajagopal [6]. The equations of motion of viscoelastic fluids are one order higher than the Navier–Stokes or boundary layer equations. Hence the boundary conditions are not sufficient to determine the solution completely. In order to overcome this difficulty, Beard and Walters [5] used a regular perturbation technique where the perturbation parameter occurs as a coefficient of highest derivative. This reduces the order of the equation. However, it solves a singular perturbation problem as a regular perturbation problem. Garg and Rajagopal [6] overcome this difficulty by augumenting the boundary conditions at infinity and used quasilinearization technique along with orthonormalization. This method has been used also by Seshadri et al. [7] to study the unsteady three-dimensional stagnation point flow of a viscoelastic fluid.

The aim of this paper is to investigate the steady mixed convection in stagnation flows of a viscoelastic fluid adjacent to a vertical surface. The paper is, in fact, the extension of the work by Ramachandran et al. [8] for Newtonian fluids and of Lok et al. [9] to viscoelastic fluids. Mixed convection in stagnation flows becomes important when the buoyancy forces, due to the temperature difference between the wall and the free stream, become high and thereby modify the flow and the thermal fields significantly. In such flows, the flow and thermal fields are no

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longer symmetric with respect to the stagnation line. Moreover, the local heat transfer rate and the local shear stress can be significantly enhanced or diminished in comparison to the pure forced convection case. Using similarity variables, the governing boundary layer equations are transformed into a system of two non-linear ordinary differential equations which are then solved analytically using the homotopy analysis method (HAM) [10–30] for both assisting and opposing flows cases. Representative results for the shear stress on the wall and velocity profiles are obtained for several values of the governing parameters, which are presented in tables and figures. To the best of our knowledge this problem has not been studied before and the results reported here are new.

2. Basic equations

Consider a two-dimensional stagnation flow normal to a heated vertical surface, which is placed in a viscoelastic fluid of uniform ambient temperature T_∞, as is shown in Fig. 1. It is assumed that the undisturbed free stream of velocity is U_∞ at large distances from the plate. In addition, we assume that the surface of the plate is heated to a variable temperature $T_w(x)$, where $T_w(x) > T_\infty$. The flow in the neighborhood of the stagnation line has the same characteristic irrespective of the shape of body. This flow is often referred to as the Hiemenz [31] flow. In the absence of heat generation and viscous dissipation, under the Boussinesq approximation and for steady state flow conditions, the boundary layer equations are given by, see [32]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}+k_0\left(u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}\right) \pm g\beta(T-T_\infty), \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

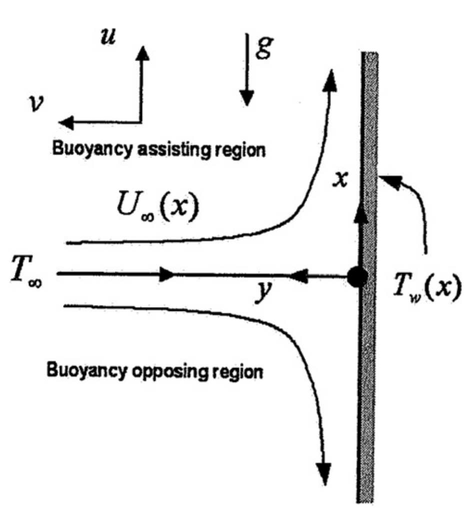

Fig. 1. Physical model and coordinate system.

subject to the boundary conditions

$$u=0,\quad v=0,\quad T=T_w(x)\quad \text{at}\quad y=0,$$
$$u\to U_\infty(x),\quad \frac{\partial u}{\partial y}\to 0,\quad T\to T_\infty\quad \text{as}\quad y\to\infty, \tag{4}$$

where x and y are the Cartesian coordinates along and normal to the plate, respectively, u and v are the velocity components along x- and y-axes, respectively, ν is the kinematic viscosity, T is the temperature, g is the acceleration due to gravity, β is the thermal expansion coefficient, α is the thermal diffusivity and k_0 is the viscoelastic parameter. Further, the +sign in Eq. (2) corresponds to the assisting flow while the −sign corresponds to the opposing flow, respectively. Here $U_\infty(x)=ax$, where a is a positive constant and we assume that $T_w(x)$ varies linearly with the coordinate x, namely, $T_w(x)=T_\infty+bx$ where b is a positive constant which means that the plate is heated.

We look for a solution of Eqs. (1)–(3) of the form

$$\eta=\sqrt{\frac{a}{\nu}}y,\quad u=axf'(\eta),\quad v=-\sqrt{a\nu}f(\eta),$$
$$\theta(\eta)=T-T_\infty/T_w-T_\infty, \tag{5}$$

where primes denote differentiation with respect to η. Substituting Equation. (5), the continuity equation (1) is satisfied automatically and from Eqs. (2) and (3) we get the following ordinary differential equations

$$f'''+ff''-f'^2+1\pm\lambda\theta-K\left(f''^2-2ff'''+ff''''\right)=0, \tag{6}$$
$$\theta''+Pr(f\theta'-f'\theta)=0, \tag{7}$$

and the boundary conditions (4) become

$$f(0)=f'(0)=0,\quad \theta(0)=1,$$
$$f'(\infty)=1,\quad f''(\infty)=0,\quad \theta(\infty)=0. \tag{8}$$

Here $\lambda(\geqslant 0)$ is the constant mixed convection parameter, K $(\geqslant 0)$ is the dimensionless viscoelastic parameter and Pr is the Prandtl number, which are defined as

$$\lambda=\frac{g\beta b}{a^2}=\frac{g\beta(T_w-T_\infty)x^2/\nu^3}{U_\infty^2x^2/\nu^2}=\frac{Gr_x}{Re_x^2},\quad K=\frac{k_0a}{\rho}, \tag{9}$$

with $Gr_x=g\beta(T_w-T_\infty)x^2/\nu^3$ being the local Grashof number, $Re_x=U_\infty x/\nu$ is the local Reynolds number and ρ is the density of the fluid.

Physical quantities of interest are the skin friction coefficient C_f and the local Nusselt number Nu_x, which are defined as

$$C_f=\frac{\tau_w}{\rho U_\infty^2},\quad Nu_x=\frac{xq_w}{\alpha(T_w-T_\infty)}, \tag{10}$$

where τ_w and q_w are the wall skin friction and the heat transfer from the plate, which are given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} + k_0\left(u\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)_{y=0},$$
$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{11}$$

Using variables (5), we get

$$Re_x^{1/2}C_f = [f'' + K(3f'f'' - ff''')]_{\eta=0} = f''(0),$$
$$Re_x^{-1/2}Nu_x = -\theta'(0). \tag{12}$$

The analytical solution of the coupled non-linear system consisting of Eqs. (6)–(8) is a obtained using the HAM.

3. Solution by homotopy analysis method (HAM)

For the analytical solution of Eqs. (6)–(8) using HAM, we choose initial approximations of $f(\eta)$ and $\theta(\eta)$ and the auxiliary linear operators $\mathscr{L}_1$ and $\mathscr{L}_2$ as

$$f_0(\eta) = \eta - 1 + \exp(-\eta), \tag{13}$$
$$\theta_0(\eta) = \exp(-\eta), \tag{14}$$
$$\mathscr{L}_1(f) = \frac{d^3 f}{d\eta^3} - \frac{df}{d\eta}, \tag{15}$$
$$\mathscr{L}_2(f) = \frac{d^2 f}{d\eta^2} - f \tag{16}$$

which have the following properties

$$\mathscr{L}_1[C_1 + C_2 e^{\eta} + C_3 e^{-\eta}] = 0, \tag{17}$$
$$\mathscr{L}_2[C_4 e^{\eta} + C_5 e^{-\eta}] = 0 \tag{18}$$

in which C_i, $i = 1$–5 are arbitrary constants. If $p(\in[0,1])$ and $\hbar_1, \hbar_2$ indicate the embedding and non-zero auxiliary parameters, respectively then the zeroth-order deformation problems are of the following form

$$(1-p)\mathscr{L}_1\left[\hat{f}(\eta;p) - f_0(\eta)\right] = p\hbar_2 H_1(\eta)\mathscr{N}_1[\hat{f}(\eta;p)], \tag{19}$$
$$\hat{f}(0;p) = 0, \quad \hat{f}'(0;p) = 0, \quad \hat{f}'(\infty;p) = 1, \tag{20}$$
$$(1-p)\mathscr{L}_2\left[\widehat{\theta}(\eta;p) - \theta_0(\eta)\right] = p\hbar_2 H_2(\eta)\mathscr{N}_2[\widehat{\theta}(\eta;p), \hat{f}(\eta;p)], \tag{21}$$
$$\widehat{\theta}(0;p) = 1, \quad \widehat{\theta}(\infty;p) = 0, \tag{22}$$

in which the non-linear operators $\mathscr{N}_1$ and $\mathscr{N}_2$ are defined by

$$\mathscr{N}_1\left[\hat{f}(\eta;p)\right] = \frac{\partial^3 \hat{f}(\eta;p)}{\partial\eta^3} + \hat{f}(\eta;p)\frac{\partial^2 \hat{f}(\eta;p)}{\partial\eta^2} - \left(\frac{\partial \hat{f}(\eta;p)}{\partial\eta}\right)^2 + 1 \pm \lambda\widehat{\theta}(\eta;p) - K\left(\left(\frac{\partial^2 \hat{f}(\eta,p)}{\partial\eta^2}\right)^2 - 2\frac{\partial \hat{f}(\eta,p)}{\partial\eta}\frac{\partial^3 \hat{f}(\eta,p)}{\partial\eta^3} + \hat{f}(\eta,p)\frac{\partial^4 \hat{f}(\eta,p)}{\partial\eta^4}\right), \tag{23}$$

$$\mathscr{N}_2\left[\widehat{\theta}(\eta;p), \hat{f}(\eta;p)\right] = \frac{\partial^2 \widehat{\theta}(\eta;p)}{\partial\eta^2} + Pr\left(\hat{f}(\eta;p)\frac{\partial \widehat{\theta}(\eta;p)}{\partial\eta} - \frac{\partial \hat{f}(\eta;p)}{\partial\eta}\widehat{\theta}(\eta;p)\right) \tag{24}$$

and $H_1(\eta)$ and $H_2(\eta)$ are the base functions. For the present flow problem we take

$$H_1(\eta) = \exp(-\eta), \quad H_2(\eta) = 1.$$

Obviously for $p = 0$ and $p = 1$ we have

$$\hat{f}(\eta;0) = f_0(\eta), \quad \hat{f}(\eta;1) = f(\eta), \tag{25}$$
$$\widehat{\theta}(\eta;0) = \theta_0(\eta), \quad \widehat{\theta}(\eta;1) = \theta(\eta). \tag{26}$$

As p increases from 0 to 1, $\hat{f}(\eta;p)$ and $\hat{\theta}(\eta;p)$ vary from $f_0(\eta)$ and $\theta_0(\eta)$ to the exact solutions $f(\eta)$ and $\theta(\eta)$. Due to Taylors theorem and Eqs. (25) and (26), we can write

$$\hat{f}(\eta;p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \tag{27}$$
$$\widehat{\theta}(\eta;p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \tag{28}$$
$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{f}(\eta;p)}{\partial p^m}\Big|_{p=0}, \quad \theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \widehat{\theta}(\eta;p)}{\partial p^m}\Big|_{p=0}, \tag{29}$$

where the convergence of the series in Eqs. (27) and (28) is dependent upon $\hbar_1$ and $\hbar_2$. Assume that $\hbar_1$ and $\hbar_1$ are selected such that the series in Eqs. (27) and (28) are convergent at $p = 1$, then due to Eqs. (25) and (26) one can write

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{30}$$
$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \tag{31}$$

Differentiating the zeroth-order deformation equations (19) and (21) m times with respect to p, dividing by $m!$, and finally setting $p = 0$ we get the following mth-order deformation problems

$$\mathscr{L}_1[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_1 \mathscr{R}_m^f(\eta), \tag{32}$$
$$f_m(0) = f_m'(0) = f_m'(\infty) = 0, \tag{33}$$
$$\mathscr{L}_2[\theta_m(\eta) = \chi_m \theta_{m-1}(\eta)] = \hbar_2 \mathscr{R}_m^\theta(\eta), \tag{34}$$
$$\theta_m(0) = \theta_m(\infty) = 0, \tag{35}$$
$$\mathscr{R}_m^f(\eta) = f_{m-1}'''(\eta) + (1-\chi_m) \pm \lambda\theta_{m-1} + \sum_{k=0}^{m-1}\left[f_{m-1-k}f_k'' - f_{m-1-k}'f_k'\right] - K\sum_{k=0}^{m-1}\left[f_{m-1-k}''f_k'' - 2f_{m-1-k}'f_k''' + f_{m-1-k}f_k^{iv}\right], \tag{36}$$
$$\mathscr{R}_m^\theta(\eta) = \theta_{m-1}''(\eta) + Pr\sum_{k=0}^{m-1}\left[\theta_{m-1-k}'f_k - \theta_{m-1-k}f_k'\right], \tag{37}$$

where

$$\chi_m = \begin{vmatrix} 0, & m \leqslant 1 \\ 1, & m > 1 \end{vmatrix}. \tag{38}$$

The series solutions of Eqs. (32)–(35) up to first few order of approximations have been obtained using MATHEMATICA which are

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta) = \lim_{M\to\infty}\left[\sum_{m=0}^{M} a^0_{m,0} + \sum_{n=1}^{2M+1} \mathrm{e}^{-n\eta}\left(\sum_{m=n-1}^{2M}\sum_{k=0}^{2m+1-n} a^k_{m,n}\eta^k\right)\right], \tag{39}$$

$$\theta(\eta) = \sum_{m=0}^{\infty}\theta_m(\eta) = \lim_{M\to\infty}\left[\sum_{n=1}^{M+1}\mathrm{e}^{-n\eta}\left(\sum_{m=n-1}^{2M}\sum_{k=0}^{2m+1-n} b^k_{m,n}\eta^k\right)\right]. \tag{40}$$

The recurrence formulas for the coefficients $a^k_{m,n}$ and $b^k_{m,n}$ of $f_m(\eta)$ and $\theta_m(\eta)$ are obtained when $m \geqslant 1, 0 \leqslant n \leqslant 2m+1$ and $0 \leqslant k \leqslant 2m+1-n$ as

$$a^0_{m,0} = \chi_m\chi_{2m+1}a^0_{m-1,0} - \sum_{q=0}^{2m}\Delta^q_{m,1}\mu^q_{1,1} - \sum_{n=2}^{2m+1}\sum_{q=1}^{2m+1-n}\Delta^q_{m,n} = \left\{\mu^q_{n,1} - (n-1)\mu^q_{n,0}\right\},$$

$$a^k_{m,0} = \chi_m\chi_{2m+1-k}a^k_{m-1,0}, \quad 0 \leqslant k \leqslant 2m+1,$$

$$a^0_{m,1} = \chi_m\chi_{2m}a^0_{m-1,1} + \sum_{q=0}^{2m}\Delta^q_{m,1}\mu^q_{1,1} - \sum_{n=2}^{2m+1}\left[n\Delta^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2m+1-n}\Delta^q_{m,n}\left(n\mu^q_{n,0} - \mu^q_{n,1}\right)\right],$$

$$a^k_{m,1} = \chi_m\chi_{2m-k}a^k_{m-1,1} - \sum_{q=k-1}^{2m}\Delta^q_{m,1}\mu^q_{1,k}, \quad 1 \leqslant k \leqslant 2m+1,$$

$$a^k_{m,n} = \chi_m\chi_{2m+1-n-k}a^k_{m-1,n} - \sum_{q=k}^{2m+1-n}\Delta^q_{m,n}\mu^q_{n,k}, \quad 2 \leqslant n \leqslant m+1, 0 \leqslant k \leqslant 2m+1-n,$$

$$b^k_{m,0} = \chi_m\chi_{2m+1-k}b^k_{m-1,0} + \sum_{q=0}^{2m+1}\Gamma^q_{m,0}\mu 1^q_{1,0},$$

$$b^0_{m,1} = \chi_m\chi_{2m}b^0_{m-1,1} - \sum_{q=0}^{2m+1}\Gamma^q_{m,0}\mu 1^q_{0,0} - \sum_{n=2}^{m+1}\sum_{q=0}^{2m+1-n}\Gamma^q_{m,n}\mu 1^q_{n,0},$$

$$b^k_{m,1} = \chi_m\chi_{2m-k}b^k_{m-1,1} + \sum_{q=k-1}^{2m}\Gamma^q_{m,1}\mu 1^q_{1,k}, \quad 1 \leqslant k \leqslant 2m+1,$$

$$b^k_{m,n} = \chi_m\chi_{2m+1-n-k}b^k_{m-1,n} + \sum_{q=k}^{2m+1-n}\Gamma^q_{m,n}\mu 1^q_{n,k}, \quad 2 \leqslant n \leqslant m+1, \quad 0 \leqslant k \leqslant 2m+1-n,$$

$$\mu^q_{1,k} = \sum_{p=0}^{q+1-k}\frac{q!}{k!2^{q+1-k-p}}, \quad q \geqslant 0, \quad 1 \leqslant k \leqslant q+1, \tag{41}$$

$$\mu^q_{n,k} = \sum_{r=0}^{q-k}\sum_{p=0}^{q-k-r}\frac{-q!}{k!(n-1)^{q+1-k-r-p}n^{r+1}(n+1)^{p+1}}, \quad q \geqslant 0, \quad 1 \leqslant k \leqslant q, \quad n \geqslant 2, \tag{42}$$

$$\mu 1^q_{1,k} = \frac{(-1)^{2q+1-2k-p}q!}{2^{q+2-k}k!}, \quad q \geqslant 0, \quad 0 \leqslant k \leqslant q+1, \tag{43}$$

$$\mu 1^q_{n,k} = \sum_{p=0}^{q+1-k}\frac{q!}{k!(n+1)^{p+1}(n-1)^{q+1-k-p}}, \quad q \geqslant 0, \quad 1 \leqslant k \leqslant q, \quad n \geqslant 2, \tag{44}$$

$$\Delta^q_{m,n} = \hbar_1\left[\begin{array}{c}\chi_{2m-n-q+1}\mathrm{e}^q_{m-1,n} \pm \lambda b^q_{m-1,1} + \chi_{2m-n-q+2}\left(\alpha^q_{m,n} - \beta^q_{m,n}\right) \\ -K\chi_{2m-n-q+2}\left(\gamma^q_{m,n} - \delta^q_{m,n} + \omega^q_{m,n}\right)\end{array}\right], \tag{45}$$

$$\Gamma^q_{m,n} = \hbar_2\left[\chi_{2m-n-q+1}g^q_{m-1,n} + Pr\chi_{2m-n-q+2}\left(\Theta^q_{m,n} - \Pi^q_{m,n}\right)\right], \tag{46}$$

$$\alpha^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-2m+2k\}}^{\min\{n,2k+1\}}\sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} d^i_{k,j}a^{q-i}_{m-1-k,n-j},$$

$$\beta^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-2m+2k\}}^{\min\{n,2k+1\}}\sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} c^i_{k,j}c^{q-i}_{m-1-k,n-j},$$

$$\gamma^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-2m+2k\}}^{\min\{n,2k+1\}}\sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} d^i_{k,j}d^{q-i}_{m-1-k,n-j},$$

$$\delta^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-2m+2k\}}^{\min\{n,2k+1\}}\sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} c^i_{k,j}a^{q-i}_{m-1-k,n-j},$$

$$\omega^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-2m+2k\}}^{\min\{n,2k+1\}}\sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} f^i_{k,j}a^{q-i}_{m-1-k,n-j},$$

$$\Theta^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-2m+2k\}}^{\min\{n,2k+1\}}\sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} a^i_{k,j}g^{q-i}_{m-1-k,n-j},$$

$$\Pi^q_{m,n} = \sum_{k=0}^{m-1}\sum_{j=\max\{0,n-m+k\}}^{\min\{n,k+1\}}\sum_{i=\max\{0,q-2m+2k+1+n-j\}}^{\min\{q,2k+1-j\}} c^i_{k,j}b^{q-i}_{m-1-k,n-j},$$

$$c^k_{m,n} = (k+1)a^{k+1}_{m,n} - na^k_{m,n}, \quad d^k_{m,n} = (k+1)c^{k+1}_{m,n} - nc^k_{m,n},$$
$$e^k_{m,n} = (k+1)d^{k+1}_{m,n} - nd^k_{m,n}, \quad f^k_{m,n} = (k+1)e^{k+1}_{m,n} - ne^k_{m,n},$$
$$g^k_{m,n} = (k+1)b^{k+1}_{m,n} - nb^k_{m,n}, \quad h^k_{m,n} = (k+1)g^{k+1}_{m,n} - ng^k_{m,n},$$
$$a^0_{0,0} = -1, \quad a^1_{0,0} = 1, \quad a^0_{0,1} = 1, \quad b^0_{0,0} = b^1_{0,0} = 0, \quad b^0_{0,1} = 1. \tag{47}$$

4. Convergence of the HAM solution

The convergence and rate of approximation for the HAM solution of the series (39) and (40) are strongly dependent upon the auxiliary parameters $\hbar_1$ and $\hbar_2$. Therefore, one can choose the proper values of $\hbar_1$ and $\hbar_2$ by plotting the $\hbar$-curves which ensure that the solution series (39)

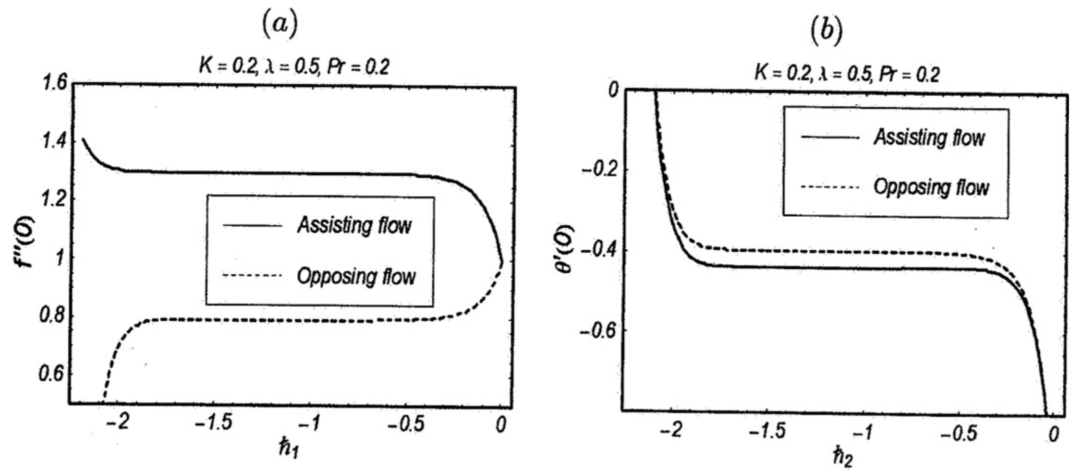

Fig. 2. $\hbar$-curves for 15th-order approximations.

Table 1
Convergence of HAM solutions for different order of approximation

Order of approximations	For assisting flow		For opposing flow	
	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
1	1.3733	0.2400	0.8500	0.2875
5	1.4133	0.3600	0.8338	0.3285
10	1.4188	0.3358	0.8197	0.3019
14	1.4190	0.3354	0.8182	0.3006
20	1.4190	0.3354	0.8176	0.3002
22	1.4190	0.3354	0.8174	0.3001
25	1.4190	0.3354	0.8174	0.3001
30	1.4190	0.3354	0.8174	0.3001

and (40) converge, as suggested by Liao [10]. For this purpose the $\hbar$-curves are plotted for 15th-order of approximations in Fig. 2 for both cases of assisting and opposing flows. Fig. 2 clearly depicts that the range for the admissible values of $\hbar_1$ and $\hbar_2$ are $-1.9 \leqslant \hbar_1 \leqslant -0.3$ and $-1.8 \leqslant \hbar_2 \leqslant -0.3$. Obviously our calculations show that the series (39) and (40) converge in the whole region of η when $\hbar_{1,2} = \hbar = -1$. Table 1 is made to show the convergence of the HAM solutions for different order of approximations.

5. Results and discussion

In order to access the influence of some physical parameters on the velocity and temperature profiles, the Figs. 2–4 are plotted. The variations of viscoelastic parameter K and the Prandtl number Pr on the velocity f' and temperature θ are shown in both the cases of assisting and opposing flows, respectively.

Fig. 3 shows the velocity f' for various values of viscoelastic parameter K in assisting and opposing flows. This Figure indicates that f' is a decreasing function of K. But the decrease in f' is slightly larger in case of assisting flow when compared with opposing flow. The boundary layer thickness is increased as K increases in both assisting and opposing flows. Fig. 4 depicts the variation of K on the temperature field θ. It can be seen that the temperature θ increases with an increase in K. Therefore one can see that

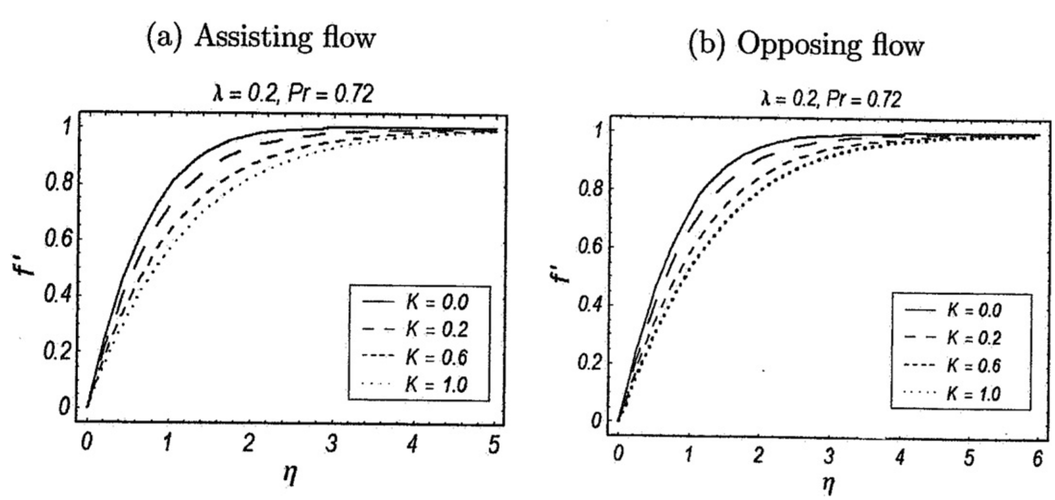

Fig. 3. Effects of viscoelastic parameter K on the velocity f' at $\hbar = -1$.

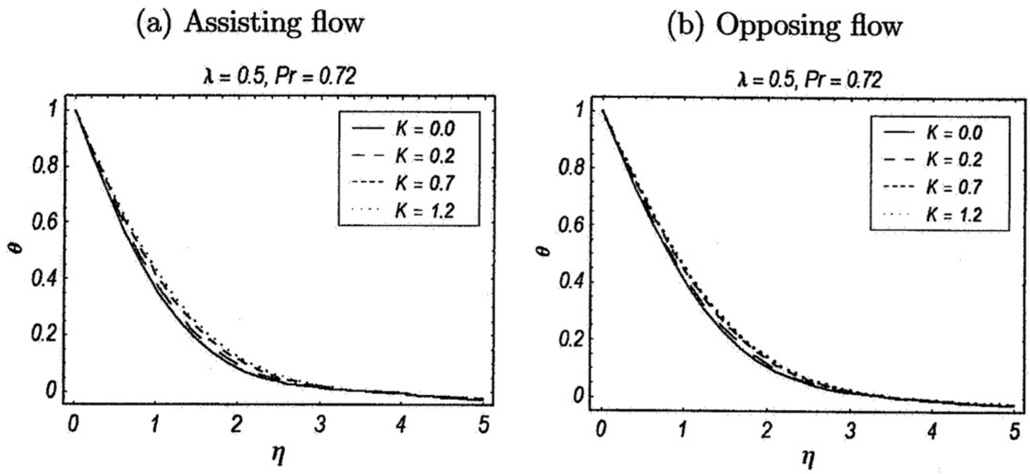

Fig. 4. Effects of viscoelastic parameter K on the temperature θ at $\hbar = -1$.

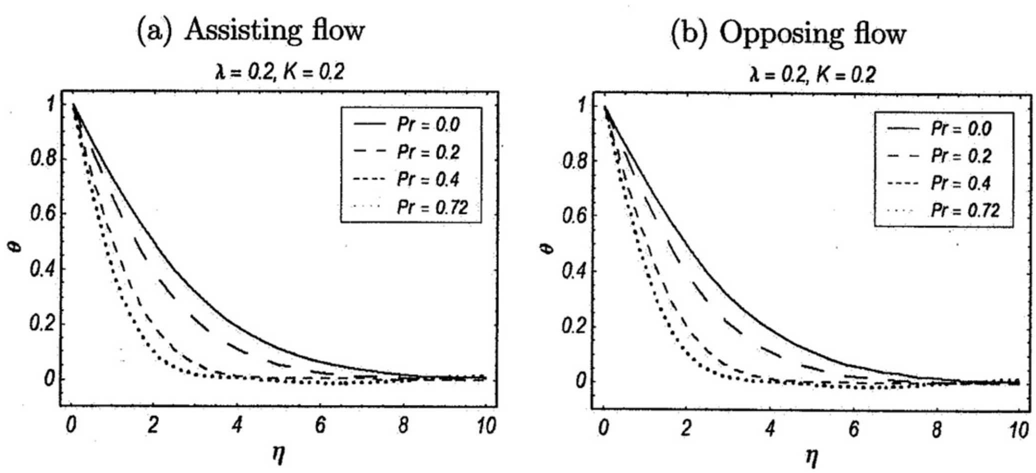

Fig. 5. Effects of Prandtl number Pr on the temperature θ at $\hbar = -1$.

influence of K on f' and θ is opposite. Also, the increase in Fig. 4 is larger when assisting flow is taken into account. Further, the thermal boundary layer thickness increases by increasing K. The influence of the Prandtl number Pr on θ is shown in Fig. 5. It is observed that θ decreases when Pr increases in the both cases. The thermal boundary layer thickness is decreased for large values of Pr.

Table 2 is made just to show the numerical values of skin friction coefficient and surface heat transfer for various values of viscoelastic parameter K and Prandtl number Pr in both cases of assisting and opposing flows, respectively. It is observed that the skin friction coefficient and local heat transfer (Nusselt number) are decreased with an increase in K in both ill assisting and opposing flows. But this increase in both $f''(0)$ and $\theta'(0)$ is larger in case of assisting flow. It is also noted that the skin friction decreases and local heat transfer increases when Pr increases in the case of assisting flow. But in case of opposing flow, the skin friction coefficient and local heat transfer increase by increasing Pr. The change in values of the skin friction coefficient and local heat transfer is larger when Pr increases in assisting flow.

Table 2
Values of skin friction $f'(0)$ and local Nusselt number Nu_x for different parameters.

λ	K	Pr	Assisting flow		Opposing flow	
			$f''(0)$	$\theta'(0)$	$f''(0)$	$-\theta'(0)$
0.2	0.0	0.2	1.3543	0.4420	1.1072	0.4235
	0.2		1.1559	0.4261	0.9558	0.4094
	0.5		0.9821	0.4097	0.8184	0.3939
	0.7		0.9044	0.4018	0.7555	0.3875
	1.0		0.8174	0.3920	0.6844	0.3785
	1.5		0.7171	0.3793	0.6015	0.3667
	2.0		0.6474	0.3698	0.5435	0.3578
	0.2	0.0	1.1933	0.0322	0.9144	0.0322
		0.2	1.1559	0.4261	0.9558	0.4094
		0.5	1.1439	0.6082	0.9689	0.5874
		0.7	1.1394	0.6903	0.9734	0.6678
		1.0	1.1353	0.7876	0.9783	0.7669

6. Final remarks

The main goal of this article is provide an analytic solution to a non-linear problem. In this work, HAM analysis has been performed for convection flow in a second-grade fluid. The simple and convenient expressions for velocity and temperature have been developed. The validity of the solutions for velocity and temperature has been explicitly discussed. The obtained series solutions confirm the power and ability of the HAM as an easy tool for computing the solution of a non-linear problem. It is noted that the negative values of the temperature gradient provide an indication of the physical fact that the heat flows from the surface to the ambient fluid. The significance of the various other parameters on the flow and temperature is highlighted. The analytic technique employed in this paper can be used to other nonlinear problems in the similar way.

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